§5

5.1 Gieseker spaces

$\mathbb{P}^2 = \mathbb{A}^2 \sqcup l_\infty$

$M(r,n) =$ moduli space of framed torsion free sheaves (E, φ)

$\varphi : E|_{l_\infty} \cong \mathcal{O}_{l_\infty}^{\oplus r}$

rank $E = r$, $c_2(E) = n$

- smooth, dimension $= 2rn$
- $M(1,n) = (\mathbb{C}^2)^{[n]}$

$\mathcal{U}^n_{SL(r)}$ = Uhlenbeck space for $G = SL(r)$, $c_2 = n$ (Sasha's lectures)

Fact. $M(r,n) \longrightarrow \mathcal{U}^n_{SL(r)}$ symplectic resolution of singularities

- cf. J. Li, Morgan for projective case
- cf. Laumon resolution $\rightarrow$ QMaps to flag of type A

$\mathbb{T} = T^2 \times T^r \curvearrowright M(r,n)$

T^2: action on $\mathbb{A}^2$ $\quad$ T^r: change of framings

5.2

Consider $\mathbb{H} := \bigoplus_n H^*_{\mathbb{T}}(M(r,n)) \cong \bigoplus_n H^{\mathbb{T}}_{4rn-*}(M(r,n))$

$\uparrow$ module over $H^*_{\mathbb{T}}(pt) = \mathbb{C}[\underbrace{\varepsilon_1,\varepsilon_2}_{T^2}, \underbrace{a_1,\cdots,a_r}_{T^r}]$

Two natural operators:

a) $\mathcal{E}$: universal sheaf on $M(r,n)\times\mathbb{P}^2$

$\mathcal{V} = R^1 p_{1*}(\mathcal{E}(-l_\infty))$ is a rank n vector b'dle over $M(r,n)$

$\Longrightarrow$ mult. of $c_i(\mathcal{V}) \curvearrowright \mathbb{H}$ $\quad$ ($r=1$ $\quad \mathcal{V}_I = \mathbb{C}[x,y]/I$)

b) $M(r,n,n+1) = \{(E_1,E_2,\varphi) \mid c_2(E_1)=n, c_2(E_2)=n+1$, $E_1 \supset E_2$ isom. on l_∞, framing compatible $\} / \cong$

$$\begin{array}{ccc} & M(r,n,n+1) & \\ \swarrow p_1 & & \searrow p_2 \\ M(r,n) & & M(r,n+1) \end{array}$$

<u>Prop</u> (1) $M(r,n,n+1)$ smooth of $\dim = 2rn + r + 1$

(2) p_2 is proper

<u>Ex</u> $r=1$

$M(1,n+1) = (\mathbb{C}^2)^{[n+1]} \ni I_{Z_2}$

$M(1,n) \quad = (\mathbb{C}^2)^{[n]} \ni I_{Z_1}$

$Z_1 \subset Z_2$ $\quad$ Z_2 is obtained from Z_1 by adding one point generically.

☆ $M(r,n,n+k)$ can be defined as in $r=1$ case, but <u>not smooth</u>.

Now $H^*_{\mathbb{T}}(M(r,n)) \xrightarrow[p_{2*}p_1^*(\cdot)]{} H^*_{\mathbb{T}}(M(r,n+1))$

For the opposite direction, we consider $M(r,n,n+1)_0 \subset M(r,n,n+1)$ $\quad \mathrm{Supp}\, E_1/E_2 = \{0\}$

$\Rightarrow p_1|_{M(r,n,n+1)_0}$ is proper $\therefore p_{1*}p_2^*(\ )$ is well-defined.

<u>Th</u> (Maulik-Okounkov, Schiffmann-Vasserot)

These operators gives a structure of a representation of the W-algebra $W(\mathfrak{gl}_r)$. motivated by AGT (Alday Gaiotto Tachikawa)

I do <u>not</u> make the statement in a precise form.

Today I only study the case $r=1$. (earlier by [Lehn]) And furthermore I set

$$\varepsilon_1+\varepsilon_2=0,$$

which means that I restrict $T^2 \supset \mathbb{C}^* \cong \{(t,t^{-1})\}$.

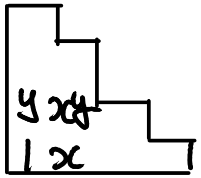
☆ $W(\mathfrak{gl}_1) \cong \infty^{\dim}$ Heis. But we also construct a Virasoro action.
It is a key ingredient in the construction (for $r=2$).

Even in these assumptions, we can still see interesting representation theory.

5.3 study of fixed points

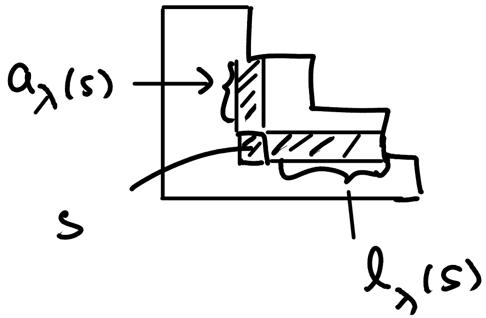
Write $X^{[n]}$ instead of $(\mathbb{C}^2)^{[n]}$ hereafter.

Prop (1) $(X^{[n]})^{\mathbb{C}^*} = \{ I_\lambda \mid$ monomial ideal corresp. to a partition $\lambda \vdash n \}$



(2) $$\mathrm{ch}_{\mathbb{C}^*} T_{I_\lambda} X^{[n]} = \sum_{s \in \lambda} t^{h_\lambda(s)} + t^{-h_\lambda(s)}$$

where $h_\lambda(s)$ is the hook length of the box $s \in \lambda$.



$h_\lambda(s) = a_\lambda(s) + l_\lambda(s) + 1$

☹ $(X^{[n]})^{T^2} \Rightarrow$ obvious $t \cdot x^l y^m = t_1^l t_2^m x^l y^m$

Compute $ch_{T^2} T_{I_\lambda} X^{[n]}$ and set $t_1 \cdot t_2 = 1$.

Then no term vanishes $\Rightarrow$ (1) & (2) follow. //

$$\therefore\ e(T_\lambda X^{[n]}) = (-1)^n \varepsilon^{2n} h(\lambda)^2 \qquad h(\lambda) := \prod_{s \in \lambda} h_\lambda(s)$$

<u>Rem</u> $h(\lambda)$ appears in the representation theory of $\mathfrak{S}_n$

$$\frac{n!}{h(\lambda)} = \text{dim irr. rep. corresponding to the partition } \lambda.$$

[Macdonald I.(7.6) & §5.Ex.2]

It is natural to consider

$$S_\lambda := \frac{1}{\varepsilon^n h(\lambda)} [I_\lambda] \in H_*^{\mathbb{C}^*}(X^{[n]}) \otimes_{\mathbb{C}[\varepsilon]} \mathbb{C}(\varepsilon)$$

normalized fixed point class.

$$i_\lambda : \{\lambda\} \to X^{[n]} \qquad [\lambda] = i_{\lambda *}[\lambda]$$

$$(-1)^n \int_{X^{[n]}} s_\lambda \cup s_\lambda = (-1)^n \frac{i_\lambda^*(s_\lambda \cup s_\lambda)}{e(T_\lambda X^{[n]})} = \frac{\cancel{(-1)^n}\, i_\lambda^* s_\lambda \cup i_\lambda^* s_\lambda}{\cancel{(-1)^n}\, \varepsilon^{2n} h(\lambda)^2} = 1$$

$$\left(i_\lambda^* s_\lambda = \frac{1}{\varepsilon^n h(\lambda)} i_\lambda^* i_{\lambda *}[\lambda] = (-1)^n \varepsilon^n h(\lambda)[\lambda] \right)$$

So $\{s_\lambda\}$: o.n.b. for $(-1)^n \int_{X^{[n]}} \cdot \cup \cdot$

Let us define an isomorphism of vector spaces with inner products :

$$\bigoplus_n H_*^{\mathbb{C}^*}(X^{[n]}) \otimes_{\mathbb{C}[\varepsilon]} \mathbb{C}(\varepsilon) \cong \mathbb{C}(\varepsilon) \otimes \Lambda^{\varepsilon} \text{ symmetric polynomials}$$

$$\text{} \qquad s_\lambda \longmapsto s_\lambda : \text{Schur function}$$

$$(-1)^n \int_{X^{[n]}} \cdot \cup \cdot \longleftrightarrow \text{standard inner product on } \Lambda$$

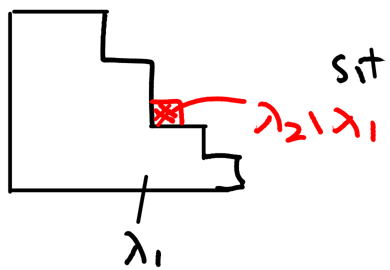
<u>Rem</u>. $H_*^{\mathbb{C}^*}(X^{[n]}) \otimes_{\mathbb{C}[\varepsilon]} \mathbb{C}(\varepsilon) \cong H_*^{\mathbb{C}^*}(\text{fixed pts}) \otimes_{\mathbb{C}[\varepsilon]} \mathbb{C}(\varepsilon)$ (localization)

Let us study the operator given by $M(1,n,n+1) = X^{[n,n+1]}$

$$X^{[n,n+1]} = \{ (I_1, I_2) \in X^{[n]} \times X^{[n+1]} \mid I_1 \supset I_2 \} \quad 2n+2 \text{ dim}$$

$\circlearrowleft \mathbb{C}^*$

A fixed pt is a pair (λ_1, λ_2) of Young diagrams



s.t. λ_2 is obtained from λ_1 by adding a box

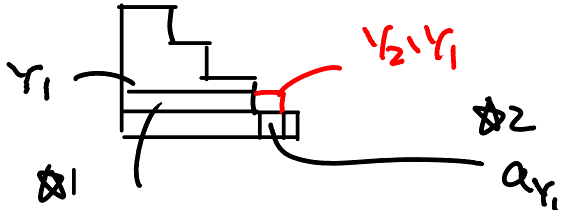
<u>Prop</u> $\mathrm{ch}\, T_{(\lambda_1,\lambda_2)} X^{[n,n+1]}$

$$= t + t^{-1} + \sum_{s\in\lambda_1} t^{-l_{\lambda_2}(s) - a_{\lambda_1}(s) - 1} + t^{l_{\lambda_1}(s) + a_{\lambda_2}(s) + 1}$$

$\mathbb{C}^2 \circlearrowleft$

<u>Cor.</u> $e(T_{(Y_1,Y_2)} X^{[n,n+1]}) = (-1)^{n+1} \varepsilon^{2(n+1)} h(Y_1) h(Y_2)$

An interesting combinatorics behind :



$a_{\gamma_1}(s) = a_{\gamma_2}(s) - 1$

$l_{\gamma_1}(s) = l_{\gamma_2}(s) - 1$

$$\prod_{s\in\lambda_1}\left(l_{\lambda_2}(s)+a_{\lambda_1}(s)+1\right) = \prod_{s\notin\boxtimes 1} h_{\lambda_1}(s) \times \prod_{s\in\boxtimes 1}\overbrace{\left(h_{\lambda_1}(s)+1\right)}^{h_{\lambda_2}(s)} = h(\lambda_1) \times \prod_{s\in\boxtimes 1}\frac{h_{\lambda_2}(s)}{h_{\lambda_1}(s)}$$

$$\prod_{s\in\lambda_1}\left(l_{\lambda_1}(s)+1+a_{\lambda_2}(s)\right) = \prod_{s\notin\boxtimes 2} h_{\lambda_1}(s) \times \prod_{s\in\boxtimes 2} h_{\lambda_1}(s)+1 = h(\lambda_1) \times \prod_{s\in\boxtimes 2}\frac{h_{\lambda_2}(s)}{h_{\lambda_1}(s)}$$

$$\therefore \prod_{s\in\lambda_1}\left(l_{\lambda_2}(s)+a_{\lambda_1}(s)+1\right)\prod_{s\in\lambda_1}\left(l_{\lambda_1}(s)+1+a_{\lambda_2}(s)\right)$$

$$= h(\lambda_1)^2 \prod_{s\in\boxtimes 1 \sqcup \boxtimes 2}\frac{h_{\lambda_2}(s)}{h_{\lambda_1}(s)} = h(\lambda_1)h(\lambda_2) \quad //$$

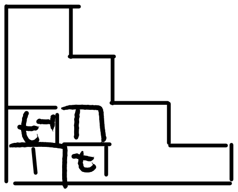
Prop. (up to sign) $[X^{[n,n+1]}] : H_*^{\mathbb{C}^*}(X^{[n]}) \to H_*^{\mathbb{C}^*}(X^{[n+1]})$
corresponds to the multiplication by e_1
(1st elementary symmetric func.)

$\because$
$$X^{[n]} \xleftarrow{p_1} X^{[n,n+1]} \xrightarrow{p_2} X^{[n+1]}$$

$$[S_{\lambda_1}] = \frac{[\lambda_1]}{e(\lambda_1)} \xrightarrow{p_1^*} e(\lambda_1)[\lambda_1] \xrightarrow{\cap X^{[n,n+1]}} \sum_{\lambda_2 \supset \lambda_1} \frac{\cancel{e(\lambda_1)}[\lambda_1]}{e(\lambda_1)e(\lambda_2)} = \sum_{\lambda_2 \supset \lambda_1} s_{\lambda_2}$$

This coincides with the Pieri formula for Schur function //

Next we study $G(V)$ $\quad V$: tautological bdle

$$\mathrm{ch}\, V\big|_\lambda =$$


$$x^i y^j \mapsto t^{i-j} x^i y^j$$

$$\therefore\ G(V)\big|_\lambda = \sum_{(i,j)=s\in\lambda} (i-j) = \sum_{s\in\lambda} c(s) = n(\lambda^t) - n(\lambda)$$
$c(s)$: content of s $\quad$ (Macdonald I.1, Ex.3

where $n(\lambda) = \sum (i-1)\lambda_i$

So it becomes a combinatorial question :

<u>Q</u>. What is the operator G on Λ: symmetric func, given by $G s_\lambda = (n(\lambda^t) - n(\lambda)) s_\lambda$?

A. G = Goulden operator

$$G := \frac{1}{2} \sum_{m,n=1}^{\infty} (\alpha_{-m}\alpha_{-n}\alpha_{m+n} + \alpha_{-m-n}\alpha_m\alpha_n)$$

up to the normalization

Goulden operator

<u>NB</u>. Mac. I.7.Ex 7 $\quad n(\lambda^t) - n(\lambda) = \frac{\chi^\lambda_\rho}{f^\lambda} h_\rho$

where $\rho = (21^{n-2})$

χ^λ_ρ : character χ^λ at the conjugacy class ρ

$f^\lambda = \chi^\lambda(1) = \dim \lambda$

$h_\rho = n!/z_\rho = n(n-1)/2$

$G \rightsquigarrow$ Virasoro algebra cf. Frenkel-Wang math.QA/0006087

$L_n = \frac{1}{n}[G, \alpha_n]$ = quadric in α_m $\Longrightarrow$ satisfies the Virasoro relations

$$[L_m, L_n] = (m-n) L_{m+n} + \delta_{m,-n} \frac{m^3-m}{12}$$

с п а с и б о